

Defining Homomorphisms and Other Generalized Morphisms of Fuzzy Relations in Monoidal Fuzzy Logics by Means of BK-Products.

Ladislav J. Kohout,

Dept. of Computer Science, Florida State University, Tallahassee, Florida 32306-4530, USA.

E-mail: kohout@cs.fsu.edu

Abstract

We generalize the previous results that were obtained by Kohout for relations based on fuzzy Basic Logic systems (BL) of Hájek and also for relational systems based on left-continuous t-norms. The present paper extends generalized morphisms into the realm of Monoidal Fuzzy Logics by first proving and then using relational inequalities over pseudo-associative BK-products of relations in these logics.

Keywords: *BK-products of relations, Generalized morphisms, Fuzzy relations, Monoidal fuzzy logics, t-norms, MV-algebras, Quantum logics, Relational inequalities, Residuated lattices, Non-associative compositions of relations.*

Invited and refereed paper presented at JCIS 2003 - 7th Joint Conf. on Information Sciences (Subsection: 9th Internat. Conf. on Fuzzy Theory and Technology), Cary, North Carolina, USA; September 2003

Contents

1	Introduction	2
2	Motivation - Crisp Generalized Morphisms	3
2.1	Crisp Standard Homomorphisms	3
2.2	Crisp Generalized Morphisms	3
2.2.1	Partial and Total Homomorphisms	3
2.2.2	Crisp Generalized Morphisms and Proteromorphisms	4
3	Solutions of Relational Inequalities Characterizing Generalized Morphisms	4
3.1	From Crisp to Fuzzy Case	5
3.1.1	Formulation of Compatibility Criteria	5
3.1.2	Algebraic Properties of BK-products of Relations	5
3.2	BK-Products of Relations	6
3.2.1	A Brief Overview of BK-Products	6
4	Residuation Bootstrap of BK-products in Monoidal Fuzzy Logics	7
4.1	Residuated Lattices and Monoidal Logics	7
4.2	The Proof of Residuation Bootstrap of BK-Products in Monoidal Fuzzy Algebras	7
5	Solutions of Inequalities in Monoidal Fuzzy Logics	8
5.1	Classification of Generalized Morphisms	8
5.2	Solutions and Proofs in Monoidal Fuzzy Algebras	9
5.3	Translation into Fuzzy Monoidal Logics	10
6	Conclusion	10

1 Introduction

Homomorphisms play an important role in mathematics, general system theory and computing as well as in large number practical applications that require comparison of structures and their matching. Many diverse problems of compatibility of structures can be unified by generalizing the concept of a homomorphism. Homomorphisms have been successfully generalized and form one of the basic concepts of mathematics of fuzzy sets. In computing and information sciences we deal to with heterogeneous relations and one way compatibilities for which both-ways commutativity of diagrams of mappings are severely inadequate. That is true for both crisp and fuzzy homomorphisms. In 1977 Bandler and Kohout introduced *generalized homomorphism*, *protohomomorphism* and *amphimorphism*, forward and backward compatibility of relations, and non-associative and pseudo-associative products (compositions) of relations in crisp setting [1]. These non-associative products were extended to fuzzy realm in 1978 [2]. The proofs in the original papers of Bandler and Kohout were based on residuation without specific use of negation [6]. Hence they are generally valid in fuzzy relational calculi based on residuated fuzzy logics. Hence the concepts of generalized morphisms, compatibility etc. can be rigorously extended to relations based on any system of fuzzy logic using continuous residuated t-norms. Rigorous proofs in the first order predicate calculus for BL family of fuzzy logics of Petr Hájek were given by Kohout [19]. Kohout [20] has also demonstrated that these relational calculi extend outside BL to the systems based on *left-continuous* t-norm family of fuzzy logics.

BL systems include the well known Gödel, Lukasiewicz and product systems of fuzzy logics [16]. Algebraically, Lukasiewicz system is an instance of MV-algebras, which also have application in development of quantum logics and measures [26]. Intuitionistic logics and linear logics play a role in theoretical computer science. All these systems are special instances of monoidal systems of fuzzy logics pioneered by Höhle. In this paper we provide proofs at the level of monoidal logics, hence our theory of many-valued logic based relations subsumes theories of relations in the above quoted systems. The general picture of the hierarchy of fuzzy logic is depicted in Figure 1.

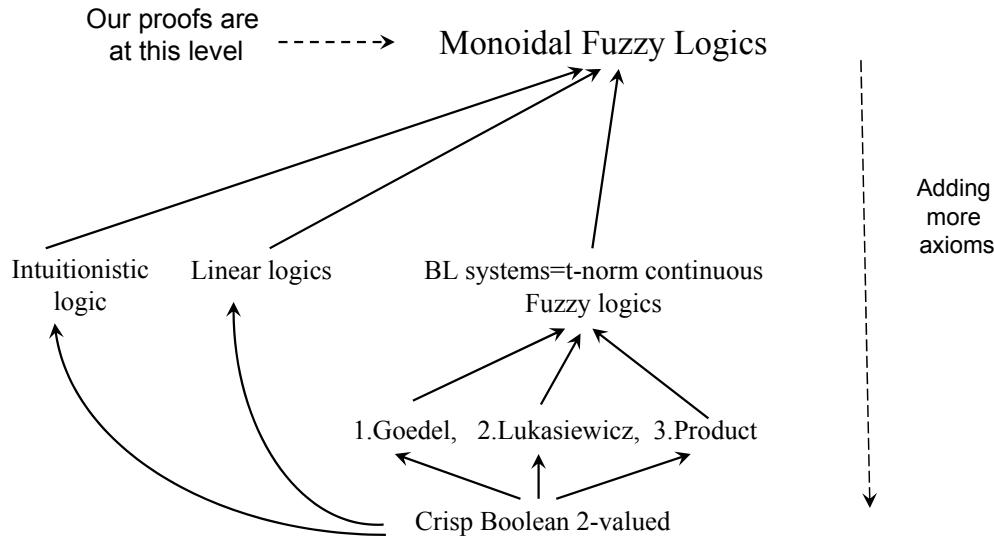


Figure 1: Hierarchy of fuzzy logics of increasing generality

2 Motivation - Crisp Generalized Morphisms

2.1 Crisp Standard Homomorphisms

Let A, B, C, D be sets with relations R, S upon them – R from A to B and S from C to D , where each relation determines some structure. In addition, we have homomorphic mappings F and G . F is from A to C and G is from B to D . There are two points of departure that stem from this fundamental algebraic notion of *homomorphism*: (i) the design or checking mappings which will “preserve” or “respect” certain given relations, and on the other hand (ii) the design or checking of relations which “absorb” or “validate” certain given mappings. For example let $A = B, C = D$ and R, S be orders. Given A and C we wish to find one or all the mappings from A to C that preserve orders – this illustrates the case (i). An example of (ii) is, given a mapping from A to C , how to match the order on A given by R , with some other order on C , or vice versa – so that some given mapping will preserve or co-preserve them. Another example is where $A = B \times B, C = D \times D$ and R and S determine some groupoids

In this situation, the conventional homomorphism yields a commuting diagram of arrows such that $R \circ G = F \circ S$, where of course, the morphisms F and G are the relations which are both covering and univalent (i.e. functional). To obtain the constructions that solve the problems (i) or (ii) requires to solve the above relational equation with respect to one of the relations R, S, F or G .

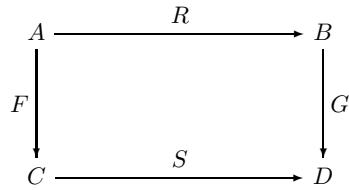


Figure 2: A diagram for conventional homomorphisms. Here homomorphic maps F and G are functions, i.e. univalent and covering relations.

When the mappings (functional relations) F and G are replaced by general relations, the equation is no longer valid but has to be replaced by two inequalities. The notion of a homomorphism splits into two independent notions, generalized morphism and generalized proteromorphism.

2.2 Crisp Generalized Morphisms

It is useful to summarize here the basic notions concerning generalized morphisms of crisp relations, as this information is not generally available in textbooks despite of the fact that generalized morphisms and proteromorphisms were introduced by Bandler and Kohout in 1977. Familiarity with the crisp equalities and inequalities characterizing these will facilitate understanding of the fuzzy case.

2.2.1 Partial and Total Homomorphisms

The following simple observation and Lemma 1 will help to comprehend the effect of relaxing equational constraints defining homomorphisms into inequalities that characterize generalized morphisms and generalized proteromorphisms.

Trivially,

$$R \circ G = F \circ S \text{ iff } F \circ S = R \circ G.$$

Composing the left hand side of the above expression with the inverse of F , the relation F^{-1} applied from the left yields $F^{-1} \circ R \circ G = F^{-1} \circ F \circ S = E \circ S = S$. Hence the equation $F^{-1} \circ R \circ G = S$ is equivalent to $R \circ G = F \circ S$.

Similarly, composing the right hand side of the above expression with the inverse of F , the relation G^{-1} applied from the right yields $F \circ S \circ G^{-1} = R \circ G \circ G^{-1} = R \circ E = R$. Hence the equation $F \circ S \circ G^{-1} = R$ is equivalent to $F \circ S = R \circ G$.

Hence, the following obvious equivalence holds: $(F^{-1} \circ R \circ G = S) \equiv (F \circ S \circ G^{-1} = R) \equiv (F \circ S = R \circ G) \equiv (R \circ G = F \circ S)$

The diagram of the Fig. 1 is a partial homomorphism if the equality above holds AND the relations F and G are *univalent* (i.e. partial functions). It is a homomorphism, if in addition both relations F and G are covering i.e. (total) functions.

This is summarized in the following obvious lemma:

Lemma 1 (Homomorphism) *For any pair of relations R and S , where S is the homomorphic image of R , the following conditions simultaneously hold:*

1. *There exist relations F and G such that the equality $(F^{-1} \circ R \circ G = S) \equiv (F \circ S \circ G^{-1} = R)$ holds, and*
2. *F and G are both univalent and covering relations.*

Lemma 2 (Partial Homomorphism) *If the arrows in Fig. 1 commute, i.e. the equality $R \circ G = F \circ S$ holds, and F and G are univalent relations, then S is a partial homomorphic image of R (i.e. partial homomorphism).*

When the relational equality $(F^{-1} \circ R \circ G = S)$ on the left, or the relational equality $F \circ S \circ G^{-1} = R$ on the right in expression (1) of Lemma 1 is replaced by the relational inclusion \sqsubseteq , the commuting diagram of Fig. 2 splits into two diagrams (see Fig. 3 below) and the notion of homomorphism has to be replaced by the notion of generalized morphisms as described in the next section.

2.2.2 Crisp Generalized Morphisms and Proteromorphisms

When the homomorphic mappings F are G not functions, the diagram of Fig. 2 does not commute any more, and the homomorphism does not exist in general case. In that case the equality $(F^{-1} \circ R \circ G = S)$ on the left, or the equality on the right in expression (1) of Lemma 1 changes into an inequality. The notion of homomorphism splits into two different notions, Generalized Morphism and Proteromorphisms. The diagrams and inequalities for these are shown in Figures 3a and 3b.

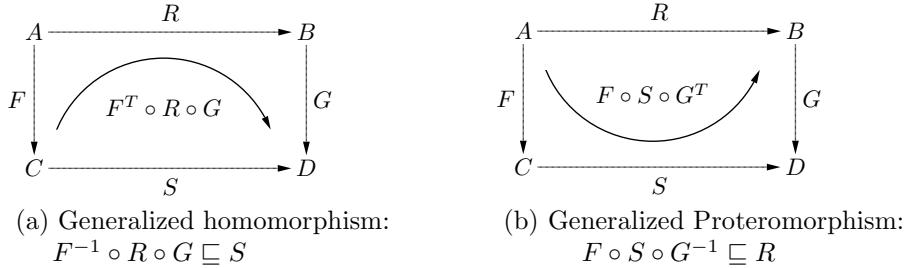


Figure 3: *Forward and Backward Compatible Relations*

3 Solutions of Relational Inequalities Characterizing Generalized Morphisms

The proofs in the original papers of Bandler and Kohout were based on residuation without specific use of negation [2]. Hence they are generally valid in fuzzy relational calculi based on residuated fuzzy logics. Hence the concepts of generalized morphisms, compatibility etc. can be rigorously extended to relations based on any system of fuzzy logic in which the implication operator is the residuum of the AND connective.

In this section we give just a sampler of selected solutions. Kohout extended the previous results of Bandler and Kohout [6] on generalized morphisms to relations based on fuzzy Basic Logic systems

(BL) of Hájek and also to relational systems based on left-continuous t-norms. In this section we give just a sampler of selected solutions for R and S .

The solutions for F and G will be presented in the sequel. Sections 4 and 5 extend generalized morphisms into the realm of Monoidal Fuzzy Logics by first proving and then using relational inequalities over pseudo-associative BK-products of relations in these logics.

3.1 From Crisp to Fuzzy Case

Relational inequalities displayed in Fig. 3 of Sec. 2.2.2 give a rigorous mathematical definition of generalized morphisms. If we want to use generalized morphisms either in pure mathematics or in applications (such as knowledge engineering, scientific computations etc.) we need some other theorems describing the properties of generalized morphisms.

For example, given any three relations chosen from R, S, F, G we may wish to compute the fourth remaining unknown one. In order to do this, we have to possess the solution of inequalities that allows us to compute the unknown relation for the known ones. Compatibility criteria provide solution for either R or S . In latter sections we shall also present the solutions for F and G .

3.1.1 Formulation of Compatibility Criteria

1. Forward Compatibility $F^{-1} \circ R \circ G \sqsubseteq S$ is fulfilled iff $R \sqsubseteq F \triangleleft S \triangleright G^{-1}$
2. Backward Compatibility $F \circ S \circ G^{-1} \sqsubseteq R$ is fulfilled iff $S \sqsubseteq F^{-1} \triangleleft R \triangleright G$
3. Bothways Compatibility is characterized by
 - (a) $F \circ S \circ G^{-1} \sqsubseteq R \sqsubseteq F \triangleleft S \triangleright G^{-1}$
or equivalently by
 - (b) $F^{-1} \circ R \circ G \sqsubseteq S \sqsubseteq F^{-1} \triangleleft R \triangleright G$

The solutions involve non-associative compositions of relations called BK-products in the literature. We shall work with the sub-product \triangleleft and super-product \triangleright . Before we proceed with further technicalities of the proofs, we briefly summarize the basic algebraic facts about \triangleleft and \triangleright .

Algebraic characterization of the interplay of the triangle sub-product \triangleleft and the triangle super-product \triangleright with the standard associative \circ (circle) product *forms the algebraic core* on which the subsequent proofs are based.

3.1.2 Algebraic Properties of BK-products of Relations

The power of both crisp and fuzzy relational calculi is substantially enhanced by introducing non-associative compositions of relations in addition to the well-known standard circle product \circ . These additional relational compositions that we called triangle and square products [23],[20],[25],[7] were first introduced by Bandler and Kohout in 1977 [1],[9],[4] and are referred to as the BK-products in the literature [15],[10],[13],[12],[14],[22].

The representational and computational power of BK-products \triangleright and \triangleleft resides in their algebraic properties. The following mixed pseudo-associativities hold for \triangleleft and \triangleright :

$$Q \triangleleft (R \triangleright S) = (Q \triangleleft R) \triangleright S,$$

$$Q \triangleleft (R \triangleleft S) = (Q \circ R) \triangleleft S$$

$$Q \triangleright (R \triangleleft S) = Q \triangleright (R \circ S).$$

The interplay of $\circ, \triangleleft, \triangleright$ that is afforded by relaxing the property of full associativity is essential for enriching the expressive power of the calculus of relations. The mutual interaction of these three relational compositions plays a crucial role in defining the key inequalities of relational calculus.

One such set of inequalities called *Residuation bootstrap of BK-products* that plays a crucial role in the development of fuzzy relational calculi [21] will be proved and used extensively in this chapter. It consists of the following relational inequalities that hold for arbitrary $V \in \mathcal{B}(\mathcal{A} \rightarrow \mathcal{C})$:

$$R \circ S \sqsubseteq V \text{ iff } R \sqsubseteq V \triangleright S^T \text{ iff } S \sqsubseteq R^T \triangleleft V$$

3.2 BK-Products of Relations

We shall briefly summarize the basic notions concerning the non-associative BK-products of relations. This knowledge is essential for fuller understanding of the proofs of the inequalities characterizing the mathematical properties of generalized morphisms that are presented in latter sections of this paper.

3.2.1 A Brief Overview of BK-Products

Mathematical definitions. Where R is a relation from X to Y , and S a relation from Y to Z , a *product relation* $R@S$ is a relation from X to Z , determined by R and S . There are several types of product used to produce product-relations [9], [23]. Each product type performs a **different logical action** on the intermediate sets, as *each logical type* of the product enforces a *distinct specific meaning* on the resulting product-relation $R@S$. In the following definitions of the products, R_{ij}, S_{jk} represent the fuzzy degrees to which the respective statements $x_i R y_j, y_j S z_k$ are true.

Table 1: \circ -product and non-associative BK-products of relations

PRODUCT TYPE	MANY-VALUED LOGIC	SET-BASED DEFINITION
Circle product:	$(R \circ S)_{ik} = \bigvee_j (R_{ij} \& S_{jk})$	$x(R \circ S)z \Leftrightarrow xR \text{ intersects } Sz$
Sub-product	$(R \triangleleft S)_{ik} = \bigwedge_j (R_{ij} \rightarrow S_{jk})$	$x(R \triangleleft S)z \Leftrightarrow xR \overset{\sim}{\subseteq} Sz$
Super-product	$(R \triangleright S)_{ik} = \bigwedge_j (R_{ij} \leftarrow S_{jk})$	$x(R \triangleright S)z \Leftrightarrow xR \overset{\sim}{\supseteq} Sz$
Square product:	$(R \square S)_{ik} = \bigwedge_j (R_{ij} \equiv S_{jk})$	$x(R \square S)z \Leftrightarrow xR \cong Sz$

There are several different notational forms in which BK-products can be expressed:

1. the notation shown in Table 1 using the concept of fuzzy set inclusion and equality [3],[4].
2. many-valued logic(MVL) based notation, which uses the logic connectives $\wedge, \&, \rightarrow$ or \equiv which is also displayed in Table 1.
3. The tensor notation (not needed in this paper).
4. The fuzzy predicate calculus form (see Table 3 in Sec. 4.1 below).

These four different forms of relational compositions are logically equivalent under some reasonable logic assumptions, producing the same mathematical results. Distinguishing these forms is, however, important when constructing fast and efficient computational algorithms [23].

The tensor notation in its presentation abstracts from the display of the type of MVL connectives shown by logic-based notation. It preserves, on the other hand, the information about the way the BK-products were composed from their components. This is important when we want to keep track of the ways in which several distinct, but logically equivalent streams of relational computation were constructed.

The logical symbols for the logic connectives AND $\&$, both *implications* and the *equivalence* in the formulas shown in Table 1 represent connectives of some many-valued logic, **chosen** according to the logic properties of the products required. An important special case is when the AND connective $\&$ is represented semantically by a t-norm $*$. If the logics are residuated, then the implications are residua of the t-norm, and the equivalence is a biresiduum of the t-norm.

The generic formula

$$(R@S)_{ik} := \oplus_j (R_{ij} \# S_{jk}),$$

yields two types of fuzzy relational products. We can replace the outer connective \oplus with \wedge (defined above) or with $\frac{1}{|J|} \sum$;

$$(R @ S)_{ik} := \bigwedge_j (R_{ij} \# S_{jk}): \text{Harsh product},$$

$$(R @ S)_{ik} := \frac{1}{|J|} \sum_j (R_{ij} \# S_{jk}): \text{Mean product}.$$

By choosing appropriate many-valued logic operations for the logic connectives, the crisp case extends to a wide variety of many-valued logic based (fuzzy) relational systems [23], [5],[7],[8],[24],[23]. While we often used in our applications the classical *min* and *max* for *t-norm* and *t-conorm*, respectively, we applied various MVL implication operators for the computation of BK-products. The details of choice of the appropriate many-valued connectives are discussed in [5],[7],[8],[24],[23].

4 Residuation Bootstrap of BK-products in Monoidal Fuzzy Logics

Now, we shall look at the ways of generalizing the Residuated Bootstrap of BK-products to monoidal fuzzy logics. It is sufficient to prove that the Residuated Bootstrap of BK-products holds in fuzzy monoidal logics. The proofs of the inequalities characterizing generalized morphisms follow then from the bootstrap inequalities in the same way as in t-norm based fuzzy logics.

4.1 Residuated Lattices and Monoidal Logics

BL systems were based on the idea that many important theorems of Zadeh's logics on $[0, 1]$ would still hold when *min* is replaced by any continuous t-norm $\&_T$ and \rightarrow_3 by the corresponding residuated \rightarrow_T implication operator. The logic systems that employ the pair a t-norm and its residuum $(\&_T, \rightarrow_T)$ as *and* and *implication* connectives were called *Basic Logics* (BL) [16]. One further extension was with left-continuous t-norms in which our Residuated Bootstrap inequalities also held. Our theorems, however, will be further generalized and shown to hold in residuated lattices. These lattices form a foundation of fuzzy logics in monoidal categories [17]. For logics, in order to possess adequate properties, complete residuated lattices are usually required.

Definition 3 *Residuated Lattice* (integral, residuated, commutative l-monoid).

A residuated lattice $\mathcal{L} = (L, \leq, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$ is a lattice containing the least element $\mathbf{0}$ and the largest element $\mathbf{1}$ and the additional two 2-argument operations \otimes and \rightarrow . \otimes is a commutative monoid for which the “residuum” \rightarrow is determined by the Galois correspondence given by the formula $a \otimes b \leq c \iff a \leq b \rightarrow c$.

The following formulas that hold in residuated lattices specified by Def. 3 will be needed in the sequel. We can see that the lattice semantics can be translated easily into first order logic formulas of fuzzy monoidal logics as shown in Table 2.

Table 3 displays the residuated lattice semantics and first order syntactic formulas of BK-products. This supplements other forms of BK-product representations that were given in Sec. 3.2.1, in Table 1.

4.2 The Proof of Residuation Bootstrap of BK-Products in Monoidal Fuzzy Algebras

Theorem 4 *Residuation bootstrap of BK-products* [21].

For arbitrary $V \in \mathcal{B}(A \rightarrow C)$,

$$T \circ U \sqsubseteq V \text{ iff } T \sqsubseteq V \triangleright U^{-1} \text{ iff } U \sqsubseteq T^{-1} \triangleleft V$$

universally holds in residuated lattice (integral, residuated, commutative l-monoid) of Def. 3.

Table 2: Lattice Semantics of the First Order Formula of Fuzzy Monoidal Logics

Lattice Semantics	First Order Logic Formulas	
$(x \otimes y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$	$(\nu \& \varphi) \rightarrow \psi = \nu \rightarrow (\varphi \rightarrow \psi)$	(1)
$(x \otimes \bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \otimes y_i)$	$\nu \& (\exists i)\varphi = (\exists i)(\nu \& \varphi)$	(2)
$(x \Rightarrow \bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \Rightarrow y_i)$	$(\nu \rightarrow (\forall i)\varphi = (\forall i)(\nu \rightarrow \varphi)$	(3)

Table 3: \circ -product and BK-products: Semantics and Syntax

PRODUCT TYPE	RESIDUATED LATTICE SEMANTICS	FIRST ORDER LOGIC FORMULAS
Circle product: $(R \circ S)$	$\bigwedge_i \bigwedge_k \bigvee_j (R_{ij} \otimes S_{jk})$	$(\forall x)(\forall z)(\exists y)(xRy \& ySz)$
Sub-product: $(R \triangleleft S)$	$\bigwedge_i \bigwedge_k \bigwedge_j (R_{ij} \Rightarrow S_{jk})$	$(\forall x)(\forall z)(\forall y)(xRy \rightarrow ySz)$
Super-product: $(R \triangleright S)$	$\bigwedge_i \bigwedge_k \bigwedge_j (R_{ij} \Leftarrow S_{jk})$	$(\forall x)(\forall z)(\forall y)(xRy \Leftarrow ySz)$
Square product: $(R \square S)$	$\bigwedge_i \bigwedge_k \bigwedge_j (R_{ij} \Leftrightarrow S_{jk})$	$(\forall x)(\forall z)(\forall y)(xRy \equiv ySz)$

Proof:

$$\begin{aligned}
 T \circ U &\sqsubseteq V \\
 \bigwedge_a \bigwedge_c (a(T \circ U)c \Rightarrow aVc) \\
 \bigwedge_a \bigwedge_c (\bigvee_b (aTb \otimes bUc) \Rightarrow aVc) \\
 \bigwedge_a \bigwedge_b \bigwedge_c (bUc \Rightarrow \bigwedge_a (bT^{-1}a \Rightarrow aVc)) \\
 \bigwedge_a \bigwedge_c (bUc \Rightarrow b(T^{-1} \triangleright V)c) \\
 U &\sqsubseteq T^{-1} \triangleright V
 \end{aligned}$$

Other parts of the formula of Th. 4 can be easily proved in a similar way.

5 Solutions of Inequalities in Monoidal Fuzzy Logics

5.1 Classification of Generalized Morphisms

Generalized Morphisms F, G from relation R to relation S are classified in Table 4.

Definition 5 (Generalized Amphimorphism) Simultaneous fulfillment of the conditions of backward compatibility and forward compatibility will be expressed as both-ways compatibility and such a morphism will be called Generalized Amphimorphism. See Fig. 4.

Homomorphism is a special kind of both-ways compatibility.

Table 4: An Overview of Generalized Morphisms

Generalized Morphism	Type of Compatibility	Relational Definition
F, G are gen. homomorphisms from R to S	$FRG: S$ are forward-compatible	$F^{-1} \circ R \circ G \sqsubseteq S$
F, G are gen. homomorphisms from R to S	$FRG: S$ are backward-compatible	$F \circ S \circ G^{-1} \sqsubseteq R$
F, G are Gen. amphimorphisms from R to S	$FRG: S$ are bothways-compatible	$F^{-1} \circ R \circ G \sqsubseteq S$ and $F \circ S \circ G^{-1} \sqsubseteq R$

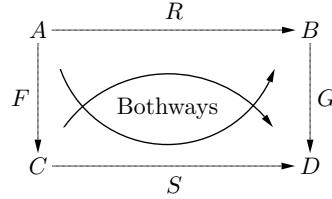


Figure 4: *Generalized Ampimorphism or Both-Ways Compatibility.*

5.2 Solutions and Proofs in Monoidal Fuzzy Algebras

In the proof we shall use the Residuation bootstrap of BK-products, namely

$$T \circ U \sqsubseteq V \text{ iff } T \sqsubseteq V \triangleright U^{-1} \text{ iff } U \sqsubseteq T^{-1} \triangleleft V$$

the validity of which in Monoidal Fuzzy Algebras we proved in the previous section (cf. Theorem 4). It will be convenient to split this expression into two parts denoting these parts as B1 and B2, respectively:

$$\mathbf{B1:} \quad T \circ U \sqsubseteq V \iff U \sqsubseteq T^{-1} \triangleleft V$$

$$\mathbf{B2:} \quad T \circ U \sqsubseteq V \iff T \sqsubseteq V \triangleright U^{-1}$$

Theorem 6 FORWARD COMPATIBILITY SOLUTION.

$$FRG : S \text{ are forward compatible} \iff F^T \circ R \circ G \sqsubseteq S \iff R \sqsubseteq F \triangleleft S \triangleright G^T$$

Proof:

Substituting $T := F^{-1}$, $U := R \circ G$, $V := S$ into B1 we obtain

$$F^{-1} \circ R \circ G \sqsubseteq S \iff R \circ G \sqsubseteq F \triangleleft S;$$

Substituting $T := R$, $U := G$, $V := F \triangleright S$ into B2 we obtain

$$R \circ G \sqsubseteq F \triangleleft S \iff R \sqsubseteq F \triangleleft S \triangleright G^{-1}; \text{ Transitivity of equivalences yields } F^{-1} \circ R \circ G \sqsubseteq S \iff R \sqsubseteq F \triangleleft S \triangleright G^{-1}. \text{ This completes the proof.}$$

Theorem 7 BACKWARD COMPATIBILITY SOLUTION.

$$FRG : S \text{ are backward compatible} \iff F \circ S \circ G^T \sqsubseteq R \iff S \sqsubseteq F^T \triangleleft R \triangleright G$$

Proof:

Substituting $T := F$, $U := S \circ G^{-1}$, $V := R$ into B1 we obtain

$$F \circ S \circ G^{-1} \sqsubseteq R \xleftarrow{B1} S \circ G^{-1} \sqsubseteq F^{-1} \triangleleft R;$$

Substituting $T := S$, $U := G^{-1}$, $V := F^{-1} \triangleright R$ into B2 we obtain

$$S \circ G^{-1} \sqsubseteq F^{-1} \triangleleft R \xleftarrow{B2} S \sqsubseteq F^{-1} \triangleleft R \triangleright G ; \text{ Transitivity of equivalences yields } F^{-1} \circ R \circ G \sqsubseteq S \iff S \sqsubseteq F^{-1} \triangleleft (R \triangleright G). \text{ This completes the proof.}$$

Theorem 8 (Forward Compatibility: Criteria for F and G)

$FRG : S$ are forward-compatible iff

$$1. F \sqsubseteq R \triangleleft (G \triangleleft S^{-1})$$

or equivalently

$$2. G \sqsubseteq R^{-1} \triangleleft (F \triangleleft S)$$

Proof:

(1): Criterion for F :

$$\begin{aligned} F^{-1} \circ R \circ G \sqsubseteq S &\iff F^{-1} \circ R \sqsubseteq S \triangleright G^{-1} \iff F^{-1} \sqsubseteq (S \triangleright G) \triangleright R^{-1} \\ &\iff F \sqsubseteq R \triangleleft (S \triangleright G^{-1})^{-1} \iff F \sqsubseteq R \triangleleft (G \triangleleft S^{-1}) \end{aligned}$$

(2): Criterion for G :

$$F^{-1} \circ R \circ G \sqsubseteq S \iff R \circ G \sqsubseteq F \triangleleft S \iff G \sqsubseteq R^{-1} \triangleleft (F \triangleright S)$$

Theorem 9 (Backward Compatibility: Criteria for F and G)

$FRG : S$ are backward-compatible iff

$$1. F \sqsubseteq (R \triangleright G) \triangleright S^{-1})$$

or equivalently

$$2. G \sqsubseteq (R^{-1} \triangleright F) \triangleright S$$

Proof:

(1): Criterion for F :

$$F \circ S \circ G^{-1} \sqsubseteq R \iff F \circ S \sqsubseteq R \triangleright G \iff F \sqsubseteq (R \triangleright G) \triangleright S^{-1}$$

(2): Criterion for G :

$$\begin{aligned} F \circ S \circ G^{-1} \sqsubseteq R &\iff (G \circ (F \circ S)^{-1})^{-1} \sqsubseteq R \iff (G \circ S^{-1} F^{-1})^{-1} \sqsubseteq R \iff \\ G \circ S^{-1} \circ F^{-1} \sqsubseteq R^{-1} &\iff G \circ S^{-1} \sqsubseteq R^{-1} \triangleright F \iff G \sqsubseteq (R^{-1} \triangleright F) \triangleright S \end{aligned}$$

5.3 Translation into Fuzzy Monoidal Logics

We have seen that the lattice semantics can be translated easily into first order logic formulas of fuzzy monoidal logics. Let us look at the translation of some important properties of residuated lattices (defined above by Def. 3) into the 1st order logic formulas. These are listed in Table 5.

More information about Monoidal Fuzzy Algebras and Monoidal Fuzzy Logics can be found in [18],[11]. Monoidal logics were originated by Ulrich Höhle.

6 Conclusion

We have focused this paper towards examining some notions and technical features of non-associative compositions of mathematical relations that are fundamental in the logic of fuzzy relations and also useful in applications. Many-valued logic based (fuzzy) extensions of relations can contribute on the theoretical side, by utilizing the elegant algebraic structure of relational systems. On the theoretical side, fuzzy relations are extensions of standard non-fuzzy (crisp) relations. By replacing the usual Boolean algebra by many-valued logic algebras, one obtains extensions that contain the classical relational theory as a special case. There is a whole spectrum of systems covered by the structures presented in this paper.

Table 5: Lattice Semantics of First Order Logic Formulas of Monoidal fuzzy Logics

Lattice Semantics	1st Order Logic Formulas	
$(x \otimes \bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \otimes y_i)$	$(x \& \exists y) = \exists y (x \& y)$	(4)
$(x \Rightarrow \bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \Rightarrow y_i)$	$(x \rightarrow \forall y) = \forall (x \rightarrow y)$	(5)
$\bigvee_{i \in I} x_i \Rightarrow y = \bigwedge_{i \in I} (x_i \Rightarrow y)$	$(\exists x \rightarrow y) = \forall (x \rightarrow y)$	(6)
$(x \otimes \bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} (x \otimes y_i)$	$(x \& \forall y) \rightarrow \forall (x \& y)$	(7)
$\bigvee_{i \in I} (x \Rightarrow y_i) \leq x \Rightarrow \bigvee_{i \in I} y_i$	$\exists (x \rightarrow y) \rightarrow (x \rightarrow \exists y)$	(8)
$\bigvee_{i \in I} (x_i \Rightarrow y) \leq \bigwedge_{i \in I} x_i \Rightarrow y$	$\exists (x \rightarrow y) \rightarrow (\forall x \rightarrow y)$	(9)

What we have done is one coherent theory which still leaves the logician leeway to choose a specific base many valued logic algebra. One consistent algebraic meta-system which leads to formulas of great variety and allows for any number of specializations. As the general algebraic structure of relations has only minimal ontological commitment, this leaves also the engineer, mathematician or scientist with choice (leeway) within which different fuzzy logics and ontologies can find elbow room. A number of different attitudes and needs can find space under this umbrella.

References

- [1] W. Bandler and L.J. Kohout. *Mathematical Relations, their Products and Generalized Morphisms*. Tech. report, Man-Machine Systems Laboratory, EES-MMS-REL 77-3, Dept. of Electrical Eng., University of Essex, Colchester, Essex, U.K., 1977. Reprinted as Ch. 2 in: Kohout, L.J. & Bandler, W., Survey of Fuzzy and Crisp Relations, Lect. Notes in Fuzzy Mathematics and Computer Sci., Creighton Univ. Omaha (to appear).
- [2] W. Bandler and L.J. Kohout. Fuzzy relational products and fuzzy implication operators. In *International Workshop on Fuzzy Reasoning Theory and Applications*, London, September 1978. Queen Mary College, University of London.
- [3] W. Bandler and L.J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980. Reprinted in: *Readings in Fuzzy Sets for Intelligent Systems*, D. Dubois, H. Prade and R. Yager (eds.), Morgan Kaufmann Publishers, San Mateo, Calif., 1993, pages 88–96.

- [4] W. Bandler and L.J. Kohout. Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems. In P.P. Wang and S.K. Chang, editors, *Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems*, pages 341–367. Plenum Press, New York and London, 1980.
- [5] W. Bandler and L.J. Kohout. Semantics of implication operators and fuzzy relational products. *Internat. Journal of Man-Machine Studies*, 12:89–116, 1980. Reprinted in Mamdani, E.H. and Gaines, B.R. eds. *Fuzzy Reasoning and its Applications*. Academic Press, London, 1981, pages 219-246.
- [6] W. Bandler and L.J. Kohout. On the general theory of relational morphisms. *International Journal of General Systems*, 13:47–66, 1986.
- [7] W. Bandler and L.J. Kohout. A survey of fuzzy relational products in their applicability to medicine and clinical psychology. In L.J. Kohout and W. Bandler, editors, *Knowledge Representation in Medicine and Clinical Behavioural Science*, pages 107–118. Gordon and Breach Publ., London and New York, 1986.
- [8] W. Bandler and L.J. Kohout. Fuzzy implication operators. In M.G. Singh, editor, *Systems and Control Encyclopedia*, pages 1806–1810. Pergamon Press, Oxford, 1987.
- [9] W. Bandler and L.J. Kohout. Relations, mathematical. In M.G. Singh, editor, *Systems and Control Encyclopedia*, pages 4000 – 4008. Pergamon Press, Oxford, 1987.
- [10] R. Bělohlávek. Similarity relations and BK-relational products. *Information Sciences*, 126(1–4):287–295, 2000.
- [11] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic/Plenum Press, New York, 2002.
- [12] B. DeBaets and E. Kerre. Fuzzy relational compositions. *Fuzzy Sets and Systems*, 60(1):109–120, 1993.
- [13] B. DeBaets and E. Kerre. A revision of Bandler-Kohout composition of relations. *Mathematica Pannonica*, 4:59–78, 1993.
- [14] B. DeBaets and E. Kerre. The cutting of compositions. *Fuzzy Sets and Systems*, 62(3):295–310, 1994.
- [15] P. Hájek. A remark on Bandler-Kohout products of relations. *Internat. Journal of General Systems*, 25(2):165–166, 1996.
- [16] P. Hájek. *Metamathematics of Fuzzy Logics*. Kluwer, Dordrecht, 1998.
- [17] U. Höhle. Monoidal closed categories, weak topoi and generalized logics. *Fuzzy Sets and Systems*, 40:15–35, 1991.
- [18] U. Höhle. Commutative residuated monoids. In U. Höhle and E.P. Klement, editors, *Non-Classical Logics and their Applications to Fuzzy Subsets: A Handbook of Mathematical Foundations of Fuzzy Set Theory*, chapter 4, pages 53–106. Kluwer, Boston, 1995.
- [19] L.J. Kohout. Generalized morphisms in BL-logics. In *Logic Colloquium 1998(The 1998 ASL European Summer Meeting, Prague)*. Association for Symbolic Logic, August 9-15 1998. (Extended abstract presenting the main mathematical theorems published the Bulletin of ASL No. 1, March 1999, pp. 116-117.).
- [20] L.J. Kohout. Theory and applications of non-associative products of relations. In T. Childers, editor, *The Logica Yearbook 1997*, chapter 16, pages 152–164. Filosofia (Publisher of the Institute of Philosophy, Czech Academy of Sciences), Prague, 1998. (ISBN 80-7007-109-5).

- [21] L.J. Kohout. Basic algebra of BK products of relations in t-norm fuzzy logics. In R. Langari and J. Yen, editors, *Proc. of FUZZ-IEEE2000*, pages 599–604, Piscataway, NJ, 2000. IEEE Neural Network Council, IEEE.
- [22] L.J. Kohout. Boolean and fuzzy relations. In P.M. Pardalos and C.A. Floudas, editors, *The Encyclopedia of Optimization*, pages 189–202. Kluwer, Boston, 2001. vol.I, A-D.
- [23] L.J. Kohout and W. Bandler. Fuzzy relational products in knowledge engineering. In V. Novák et al., editor, *Fuzzy Approach to Reasoning and Decision Making*, pages 51–66. Academia and Kluwer, Prague and Dordrecht, 1992.
- [24] L.J. Kohout and M. Kallala. Choice of fuzzy optimal logics for pattern classifiers by means of measure analysis. In *Proc. of 7th International Congress on Cybernetics and Systems*, Imperial College, London, September 1987.
- [25] L.J. Kohout and E. Kim. Reasoning with cognitive structures of agents I: :Acquisition of rules for computational theory of perceptions by fuzzy relational products. In Da Ruan and E. Kerre, editors, *Fuzzy IF-THEN Rules in Computational Intelligence*, chapter 8, pages 161–188. Kluwer, Boston, 2000.
- [26] B. Riečan and T. Neubrunn. *Integral, Measure, Ordering*. Kuwer Academic Publishers, Ister Science, Dordrecht, Bratislava, 1997.